# Symmetry breaking by periodic potentials in quantum Hall systems

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## Abstract

The quantum Hall system in a high Landau level under the influence of a one dimensional spatially periodic potential is investigated by direct diagonalization. We consider the $N = 2$ Landau level at half filling for finite systems with aspect ratio one. Hence the symmetry breaking, or lack thereof, is due to the periodic potential. In agreement with mean field and perturbation theory studies, direct diagonalization indicates that for weak potentials stripes order perpendicular to the external potential. At stronger values of the potential, the potential dominates and the periodicity seen is determined by the potential and not the stripes. An example is given where randomness tends to enhance the symmetry breaking induced by the periodic potential.

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

A recent fascinating experimental discovery in quantum Hall systems is the existence of anisotropic diagonal resistivity in samples where the highest filled Landau level has index $N \geqslant 2$ and the highest filled Landau level is near half filling [1]. Physical conditions on materials for the observation of the effect include low temperature and high mobility when no magnetic field is present. A natural interpretation of these experiments is the formation of stripes in the electron density and indeed theoretical arguments for the existence of stripes in these systems was given before the above experiments were done [2]. Despite a large number of subsequent computational and theoretical investigations of 'striped' quantum Hall systems a variety of issues remain unresolved. The issue we will address in this paper is what determines the orientation of the stripes [3]. For a fixed experimental system, the orientation of the stripes (more precisely, the direction in which the diagonal resistivity is low) is robust. For example, in all the experiments discussed in the thesis of Cooper the stripes are oriented along the [110] direction of the GaAs crystal [4].

Based on STM experiments, it was suggested that this symmetry breaking is caused by a periodic potential in the $x$ direction (i.e. $A\cos(kx)$) [5]. This suggestion has motivated additional experimental and computational studies. The subsequent experimental studies indicate no clear correlation between a potential on the surface of the sample and the existence of stripes [6]. In any case, present STM experiments measure the surface of the experimental system; the two dimensional electron layer is buried a considerable distance below the surface and is invisible to STM investigation. To reiterate, modulation at the surface seems to be irrelevant to stripe orientation, however the 2d electron layer is inaccessible to experiments sensitive to a periodic potential. It therefore does not seem unreasonable to consider computational approaches. Computational studies of a 'high' Landau level subject to a periodic potential have included perturbation theory [7] and mean field theory [8] approaches. The result of both computational studies indicate that a weak periodic potential tends to orient the stripes in a rather counter intuitive way, namely perpendicular to the potential (for the electron density $\rho \sim \cos(k'y)$ if $V \sim \cos(kx)$). For strong potentials stripes orient themselves parallel (in the above sense) to the potential [7].

We will consider a complementary computational technique: direct diagonalization [9]. The advantage of this technique, as opposed to mean field theory, is that direct diagonalization is numerically exact. The disadvantage of direct diagonalization is the limitation to small system sizes. One expects mean field theory to be rather accurate for high Landau levels, however $N = 2$ is not terribly high, and the possibility of competing 'phases' due to the periodic potential could give difficulty to a mean field approach. Direct diagonalization also allows the incorporation of disorder in a straightforward, though computationally demanding, way [10].

Our paper is organized as follows: in the next section the model is defined and numerical techniques are briefly explained. In the third section we present numerical results and in the final section give conclusions and directions for future work.

## 2. The model and the numerical approach

The model considered is electrons interacting via the Coulomb interaction in the $N = 2$ Landau level. The electrons are spin polarized and electrons in lower Landau levels provide an inert background. This choice is suggested by experiment, stripes occurring in the $N = 2$ level and numerical expediency. Stripes exist in higher Landau levels as well, but in higher levels the effective interaction is more long ranged, causing presumably more severe finite size difficulties. The half filled case in the $N = 2$ Landau level will be studied because in experiment stripes are robust at this filling. Since 'striped' behaviour is sought, rectangular clusters with periodic boundary conditions are used. The numerical results presented will involve clusters with aspect ratio 1 ($a = L_x = L_y = b$). The rationale for this choice is to avoid symmetry breaking due to the cluster shape. In previous direct diagonalization studies, symmetry breaking was provided by the cluster shape; here symmetry breaking (or lack thereof) is only due to the periodic potential.

The Hamiltonian $H$ is given schematically by $H = H_i + H_v$, where $H_i$ is the electron–electron interaction term and $H_v$ is the interaction of the electrons with the potential. To get an explicit expression, in second quantized form, suitable for numerical evaluation, let us recall some basic physics of electrons in two dimensions moving in magnetic fields [11]. Firstly, the magnetic length $l$ is given by $l = \sqrt{\frac{\Phi}{2\pi B}}$ where $B$ is the magnetic field and $\Phi = \frac{hc}{e}$ is the flux quantum. The numerical 'system' consists of a rectangle of dimensions $a$ in the $x$ direction and $b$ in the $y$ direction, with the number of single particle states $m$ in the system determining

$a$ and $b$ through the relation $\frac{ab}{2\pi l^2} = m$. Single particle states, orbitals, in the Landau gauge, 'look like' stripes oriented along the $y$ direction with $x$ 'width' of order $l$. The position of the $j$th orbital in $x$ is determined by its centre $X_j = \frac{2\pi j l^2}{b}$. Finally

$$H_{\mathrm{i}} = \sum A_{j_1 j_2 j_3 j_4} a^{\dagger}_{j_1} a^{\dagger}_{j_2} a_{j_3} a_{j_4}$$

where the $j$ run from 1 to $m$ and $a^{\dagger}_j$ creates an electron in the $j$th orbital. At this point, units are used where $l = 1$ and $\frac{e^2}{\varepsilon l} = 1$. Then the coefficient $A_{j_1 j_2 j_3 j_4}$ is given by

$$A_{j_1 j_2 j_3 j_4} = \frac{1}{2ab} \sum_{q,q \neq 0} \frac{2\pi}{q} \delta(t + j_4 - j_1)' \delta(j_1 + j_2 - j_3 - j_4)' \left( L_2 \left( \frac{q^2}{2} \right) \right)^2$$
$$\times \exp\left( -\frac{q^2}{2} \right) \exp\left( -\mathrm{i} \frac{2\pi s}{m} (j_1 - j_3) \right). \tag{1}$$

Here $\vec{q} = (\frac{2\pi s}{a}, \frac{2\pi t}{b})$ $s$ and $t$ integers, $\delta(\cdots)'$ is a Kronecker delta where the argument is zero modulo $m$ and $L_2$ is the second Laguerre polynomial, $L_2(\frac{q^2}{2}) = \frac{1}{8}(q^4 - 8q^2 + 8)$. The term $H_{\mathrm{v}}$ is given for a general potential $V(r)$ with Fourier transform $V(q)$ by

$$H_{\mathrm{v}} = \sum_{l,k} f_{l,k} a^{\dagger}_l a_k$$

$$f_{l,k} = \frac{1}{ab} \sum_{\vec{q}} V(q) \exp\left( \frac{-q^2}{4} \right) L_2 \left( \frac{q^2}{2} \right) \sum_M \delta(k - l + t - Mm) \exp\left( \mathrm{i}\pi s \left( \frac{k}{m} + \frac{l}{m} + M \right) \right).$$
$$\tag{2}$$

We shall make two choices for $V(r)$, firstly $V(r)$ a periodic potential and secondly a periodic potential plus a random potential. The discussion of the random potential is deferred to the results section and we concentrate here on the purely periodic case.

For the purely periodic case, the choice we have made is $V(\vec{r}) = V(x) = 2V \cos(\frac{2\pi s}{a} x + \alpha)$, i.e. $V$ has only one Fourier component of wavelength $\lambda = \frac{a}{s}$ ($q = \frac{2\pi s}{a}$) and $\alpha = \frac{-\pi s}{m}$. Hence $f_{lk} = 2V \exp(-\frac{q^2}{4}) L_2(\frac{q^2}{2}) \cos(\alpha + \frac{2k}{m}\pi s) \delta_{lk}$. This particular choice maintains momentum conservation in the $y$ direction and is thus convenient for numerical implementation. For example, consider the case of 12 electrons in 24 orbitals, then the state space has dimension $\binom{24}{12} = 2\,704\,156$. With momentum conservation we can deal with each momentum separately, each having approximate dimension of $\frac{2.7 \times 10^6}{24} \sim 10^5$. Clearly, $100\,000$ is better than 2.7 million! The choice of $\alpha$ determines the relation of the peaks and troughs of the potential to the centres of the orbitals. Somewhat surprisingly, the ground state energy does depend on $\alpha$, though by conducting a limited number of numerical trials, we have found this dependence is weak. One possibility to choose $\alpha$ is to minimize the energy; unfortunately such a choice is inconvenient using direct diagonalization. We have made the choice $\alpha = \frac{-\pi s}{m}$ since in this case the potential is symmetric about the middle of the 'chain' of orbitals. For example, for this choice of $\alpha$, for 24 orbitals, $f_{kk} \equiv f_k = f_{25-k}$. This property can be useful for the application of density matrix renormalization group methods [12] (see the conclusion section).

After making a choice of $V$, the strength of the potential, and $s$, $a/s$ being the wavelength of the potential, the ground state energy and the wavefunction are calculated using the Lanczos method. As previously mentioned, momentum conservation in the $y$ direction is used to reduce the size of the state space one needs to treat. To determine whether stripes exist and the direction of the stripes in the ground state wavefunction, following [9, 10], we study the projected equal time density–density correlation function $(S_0(q))_p$. Here $(S_0(q))_p$ is the 'ordinary' equal time density–density correlation function $S_0(q) = \frac{1}{N_{\mathrm{e}}} \sum_{i,j} \mathrm{e}^{\mathrm{i}q \cdot (r_i - r_j)}$ in the second quantized
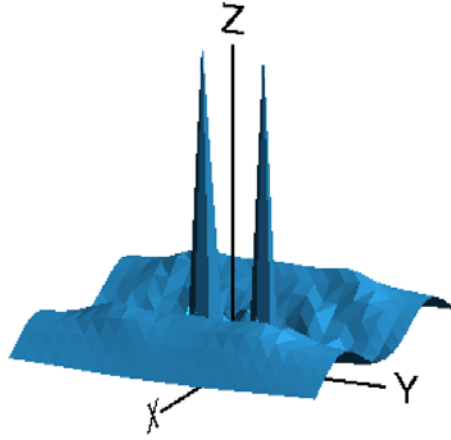
**Figure 1.** The projected equal time density–density correlation function. A system with 12 electrons in 24 states is considered, the periodic potential has strength $V = 0.05$ and wavelength $a$.

form with a factor of $\exp(\frac{-q^2}{4})(L_2(\frac{q^2}{2}))^2$ removed ($N_e$ is the number of electrons and $r_i$ is the coordinate of the $i$th electron). Explicitly

$$(S_0(q))_p = \sum B_{j_1 j_2 j_3 j_4} a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3} a_{j_4}$$

$$B_{j_1 j_2 j_3 j_4} = \frac{1}{2N_e} \delta(j_1 + j_2 - j_3 - j_4)' \left\{ \delta(t + j_4 - j_1)' \exp\left(-i\frac{2\pi s}{m}(j_1 - j_3)\right) \right.$$

$$\left. + \delta(-t + j_4 - j_1)' \exp\left(i\frac{2\pi s}{m}(j_1 - j_3)\right) \right\} \tag{3}$$

$$\vec{q} = \left(\frac{2\pi s}{a}, \frac{2\pi t}{b}\right).$$

Henceforth $(S_0(q))_p$ will be denoted $S_0(q)$. The presence of stripes is suggested by a large value of $S_0(q)$ at a particular '$q$' value and symmetry breaking is indicated by $S_0(q_x, q_y) \neq S_0(q_y, q_x)$. The discussion of numerical results is given in the following section.

## 3. Numerical results

We have studied two values of the wavelength of the potential for two system sizes for a number of potential strengths. The wavelength of the potentials studied are $\lambda = a$ and $a/5$. Our calculations and previous work [10] indicate the natural wavelength of the stripes $\lambda_s$, for the system sizes studied, is $a/2$. Hence $\lambda = a$ corresponds to a situation where $\lambda > \lambda_s$ and $\lambda = a/5$ corresponds to a case $\lambda < \lambda_s$. The system sizes studied are 10 electrons in 20 orbitals and 12 electrons in 24 orbitals. These sizes were chosen to be large enough to eliminate some finite size effects while keeping the calculations numerically tractable. Clusters were chosen so that an even number of electrons corresponds to a half filled state to encourage the formation of stripes; 12 electrons in 24 orbitals was studied as opposed to 11 electrons in 22 orbitals.

To provide orientation, let us look at $S_0(q)$ for $V = 0.05$, for $\lambda = a$, 12 electrons in 24 orbitals (figure 1). Recall for all calculations reported in this paper $N = 2$ and the cluster is square. One observes pronounced peaks, of height 3.77, at $(s, t) = (0, \pm 2)$ where $\vec{q} = (\frac{2\pi s}{a}, \frac{2\pi t}{b})$. The peak at $S_0(q = (0, 0)) = 12$ has been removed from the graph. The wavevectors in the $y$ direction indicate that the stripes point along the $x$ axis, that the stripes
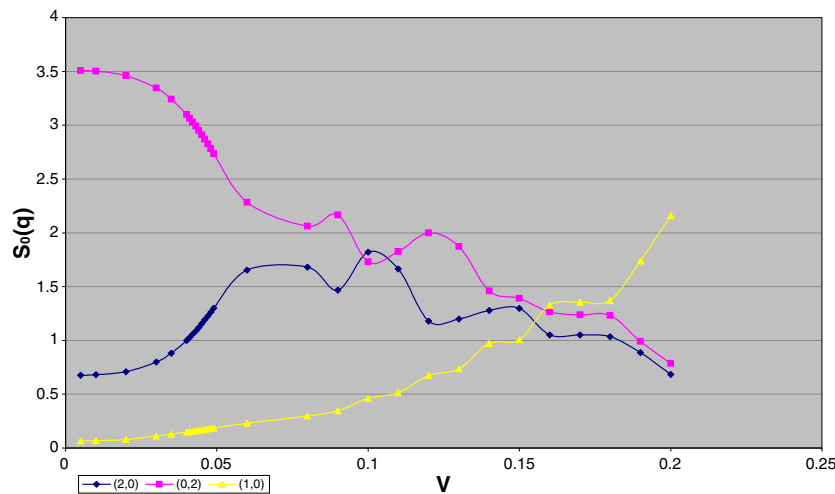
**Figure 2.** The projected equal time density–density correlation function as a function of the periodic potential strength. The system has 10 electrons in 20 states and the periodic potential has the wavelength $a$. The correlation function is evaluated at $(s, t) = (0, 2)$, $(2, 0)$ and $(1, 0)$.

are perpendicular to the imposed potential. In figure 2 we see how the (2, 0), (0, 2) and (1, 0) values of $S_0(q)$ change as one varies the strength of the periodic potential for $\lambda = a$ and 10 electrons in 20 orbitals. One observes for a weak potential that stripes perpendicular to the potential dominate with the (2, 0) value growing as $V$ increases. However, before the (2, 0) component dominates, the $q$ value $(s, t) = (1, 0)$, the wavevector of the potential, becomes largest. The structure in the graph is correlated with level crossing of the ground state. For example, near $V = 0.1$ the momentum of the ground state changes from $k_y = 5$ to 1(9), the number in parenthesis indicating the value of the momentum for a degenerate ground state. We have checked, in the cases considered, that the degenerate ground states have the same $S_0(q)$ values. It is also important to note that the lines plotted through the points are 'mere' guides for the eye. In particular, in the vicinity of level crossings the curves may be discontinuous. In figure 3 a similar plot is shown for $\lambda = a$ with 12 electrons in 24 orbitals. In agreement with the previous plot, ordering perpendicular to the potential is indicated for relatively weak coupling with $q = (2, 0)$ and (1, 0) growing as $V$ becomes large. In contrast to the 10 electron case, one needs a finite $V$ to break the symmetry between $q = (2, 0)$ and (0, 2).

Let us now turn to the case $\lambda = a/5$, $\lambda < \lambda_s$. In figure 4, we plot the (2, 0), (0, 2) and (5, 0) values of $S_0(q)$ as a function of $V$. In comparison to the $\lambda = a$ potential case, we note some similar features, $S_0(q = (0, 2))$ is largest for small $V$ and $S_0(q = (5, 0))$, the $q$ of the potential becomes largest as $V$ increases. However, for the $\lambda = a/5$ case, the (2, 0) value never becomes very large. In figure 5, a similar plot is shown for 12 electrons in 24 orbitals. Consistent with figure 4, ordering perpendicular to the potential is indicated for relatively weak $V$ with the $q = (5, 0)$ value growing as $V$ becomes large. As in the previously studied 12 electron case, one needs a finite $V$ to break the symmetry between $S_0(q)$ $q = (2, 0)$ and (0, 2). Near $V = 0.315$, we see in figure 5 a large drop in $S_0(q)$, which appears to be a discontinuity; this corresponds to a level crossing from 12(24) to 1(11).

For the $\lambda = a/5$ case a larger value of $V$ is necessary to break the $x$–$y$ symmetry and generally the dependence of $S_0(q)$ on $V$ is weaker. This can be understood qualitatively through the simple observation that when potential is projected into a Landau level, higher wavevector components are suppressed more than lower wavevector components. Explicitly, in the second
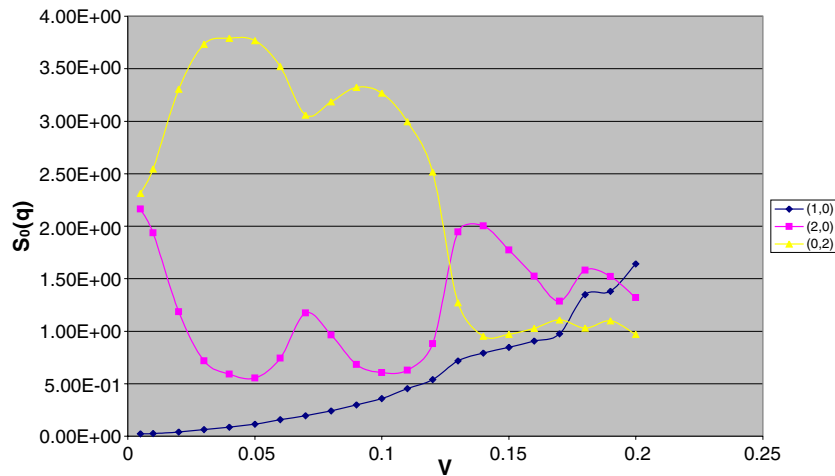
**Figure 3.** The projected equal time density–density correlation function as a function of the periodic potential strength. The system has 12 electrons in 24 states and the periodic potential has the wavelength $a$. The correlation function is evaluated at $(s, t) = (0, 2)$, $(2, 0)$ and $(1, 0)$.
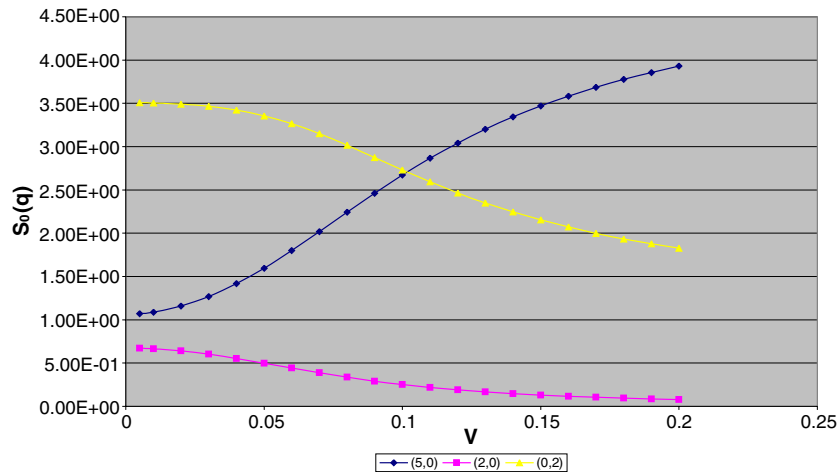


**Figure 4.** The projected equal time density–density correlation function as a function of the periodic potential strength. The system has 10 electrons in 20 states and the periodic potential has the wavelength $a/5$. The correlation function is evaluated at $(s, t) = (0, 2)$, $(2, 0)$ and $(5, 0)$.

quantized expression for the potential, $f_{lk}$ has a factor $\exp(-\frac{q^2}{4})L_2(\frac{q^2}{2})$; numerically this factor equals 0.63 for $\lambda = a$ and 0.12 for $\lambda = a/5$.

How is symmetry breaking by the periodic potential affected by disorder? In any real experimental situation disorder is present; unfortunately the introduction of disorder makes the numerics more difficult as momentum conservation is lost. As a preliminary study of the effects of disorder, we have investigated 12 electrons in 24 orbitals for a periodic potential of $\lambda = a/5$ and strength $V = 0.05$ under the added influence of a short range random potential $U(r)$. The random potential $U(r)$ is chosen to be delta correlated, i.e. $\langle U(r)U(r') \rangle = U_0 \delta(r - r')$ and we have looked at the case $U_R = 0.05 = \sqrt{\frac{3}{2}U_0}$. Averaging over 18 random configurations
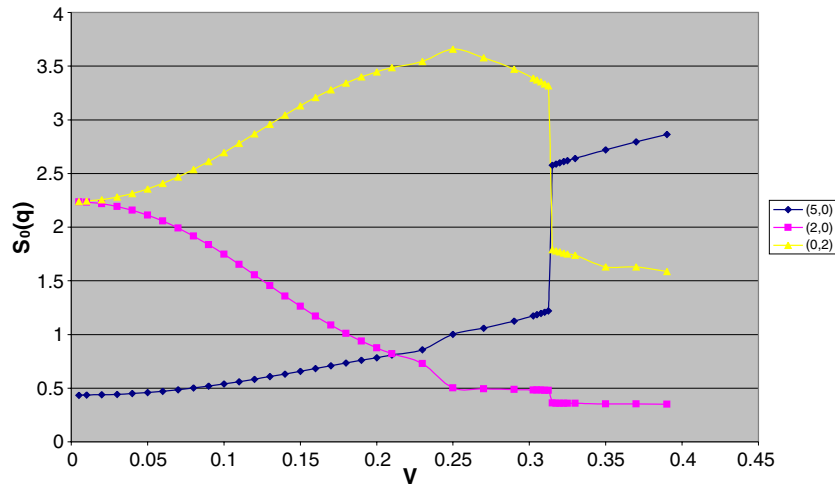
**Figure 5.** The projected equal time density–density correlation function as a function of the periodic potential strength. The system has 12 electrons in 24 states and the periodic potential has the wavelength $a/5$. The correlation function is evaluated at $(s, t) = (0, 2)$, $(2, 0)$ and $(5, 0)$.

$\langle S_0(q = (0, 2)) \rangle = 2.91$ and $\langle S_0(q = (2, 0)) \rangle = 1.49$ both with rms deviations $\sigma = 0.74$ and $\sigma/\sqrt{N} = 0.17$. We were only able to do a limited amount of averaging and parameter value exploration because each 'trial' took about 1 day on a work station. By way of comparison, without the random potential, $S_0(q = (0, 2)) = 2.36$, $S_0(q = (2, 0)) = 2.11$. Hence at least for these parameter values, randomness tends to enhance the symmetry breaking of the periodic potential.

## 4. Conclusions

In agreement with mean field and perturbation theory studies, direct diagonalization indicates that, for weak potentials, stripes order perpendicular to the external potential. Unlike [7], we do not see much evidence for ordering parallel to the potential. At stronger values of the potential, the potential dominates and the periodicity seen is determined by the potential and not the stripes. This seems particularly true for the case when the wavelength of the potential is smaller then the stripe wavelength. By comparing the behaviour of 20–24 orbitals for very weak potentials, two types of behaviour are observed. For 20 orbitals given 'epsilon' potential strength 'nice' stripes form, however for 24 orbitals a threshold of potential strength needs to be met before 'good' stripes are present. In the present calculations, due to the limited number of system sizes accessible, it is not possible to address which behaviour occurs for very large systems. A possible way to treat very large systems accurately is the recently developed density matrix renormalization group (dmrg) approach to quantum Hall systems [13]. We have done a limited number of such dmrg calculations but the algorithm we have used, which does not make use of momentum conservation in the $y$ direction, does not have sufficient accuracy to calculate $S_0(q)$ reliably. As a concrete example, we have calculated using the finite system dmrg algorithm, without momentum conservation, 12 electrons in 24 states, 100 states in the blocks, for $\lambda = a/5$ and $V = 0.05$. The energy we obtain is $-1.973$ compared to the direct diagonalization result of $-1.979$, i.e. not bad. However, the $S_0(q = (0, 2))$, $(S_0(q = (2, 0)))$ value given by dmrg is 4.08(0.48) as opposed to the numerically exact

result of 2.36(2.11). The tendency to 'oversell' striped behaviour seems to characterize our dmrg algorithm. It would be interesting in future work to implement a dmrg algorithm with momentum conservation [13] to treat large system sizes. Another interesting direction for future investigation is the effect of disorder, as well as a periodic potential on these systems. Our very preliminary results indicate that disorder tends to enhance the stripes. Again, it would be nice to be able to treat larger system sizes, however, this may be problematical if the dmrg algorithm for a two dimensional system really needs momentum conservation to be sufficiently accurate. Finally, and most importantly, does a periodic potential cause symmetry breaking in experimental systems? Recently, for example, a competing mechanism has been suggested based on electron–phonon interaction [14]. However, as emphasized in [7] whatever the 'native' cause of symmetry breaking it would be interesting to experimentally modify the two dimensional electron layer, artificially imposing a periodic potential.

## Acknowledgments

## References

[1]  Lilly M P, Cooper K B, Eisenstein J P, Pfeiffer L N and West K W 1999 *Phys. Rev. Lett.* **82** 394
      Du R R, Tsui D C, Stormer H L, Pfeiffer L N, Baldwin K W and West K W 1999 *Solid State Commun.* **109** 389
[2]  Koulakov A A, Fogler M M and Shklovskii B I 1996 *Phys. Rev. Lett.* **76** 499
      Moessner R and Chalker J T 1996 *Phys. Rev.* B **54** 5006
[3]  Zhu J, Pan W, Stormer H L, Pfeiffer L N and West K W 2003 *Phys. Rev. Lett.* **88** 116803
      Cooper K B, Eisenstein J P, Pfeiffer L N and West K W 2004 *Phys. Rev. Lett.* **92** 02686
[4]  Cooper K B 2003 *Thesis* Caltech
[5]  Willet R L, Hsu J W P, Natelson D, West K W and Pfeiffer L N 2001 *Phys. Rev. Lett.* **87** 126803
[6]  Cooper K B, Lilly M P, Eisenstein J P, Jungwirth T, Pfeiffer L N and West K W 2001 *Solid State Commun.* **119** 89
[7]  Aoyama T, Ishikawa K and Maeda N 2002 *Europhys. Lett.* **59** 444
[8]  Yoshioka D 2001 *J. Phys. Soc. Japan* **70** 2836
[9]  Rezayi E H, Haldane F D M and Yang K 1999 *Phys. Rev. Lett.* **83** 1219
      Rezayi E H, Haldane F D M and Yang K 2000 *Phys. Rev. Lett.* **85** 5396
[10]  Sheng D N, Wang Z and Friedman B 2002 *Phys. Rev.* B **66** 161103
[11]  Girvin S M 2000 The quantum Hall effect: novel excitations and broken symmetries *Topological Aspects of Low Dimensional Systems* ed A Comtet, T Jolicoeur, S Ouvry and F David (Berlin: Springer)
[12]  White S R 1993 *Phys. Rev.* B **48** 10345
[13]  Shibata N and Yoshioka D 2001 *Phys. Rev. Lett.* **86** 5755
[14]  Fil D V 2000 *Low Temp. Phys.* **26** 581